Eur. Phys. J. B **25**, 479–482 (2002)
DOI: 10.1140/epjb/e20020053

THE EUROPEAN PHYSICAL JOURNAL B
EDP Sciences

Exact propagator of the Fokker-Planck equation with a space-dependent diffusion coefficient and a time-dependent mean-reverting force

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Received 23 October 2001 and Received in final form 24 December 2001

Abstract. We have investigated the algebraic structure of the Fokker-Planck equation with a variable diffusion coefficient and a time-dependent mean-reverting force. Such a model could be useful to study the general problem of a Brownian walker with a space-dependent diffusion coefficient. We also show that this model is related to the Fokker-Planck equation with a constant diffusion coefficient and a time-dependent anharmonic potential of the form $V(x,t) = \frac{1}{2}a(t)x^2 + b\ln x$, which has been widely applied to model different physical and biological phenomena, *e.g.* the study of neuron models and stochastic resonance in monostable nonlinear oscillators. Using the Lie algebraic approach we have derived the exact diffusion propagators for the Fokker-Planck equations associated with different boundary conditions, namely (i) the case of a single absorbing barrier, and (ii) the case of two absorbing barriers. These exact diffusion propagators enable us to study the time evolution of the corresponding stochastic systems.

PACS. 05.40.-a Fluctuation phenomena, random processes, noise, and Brownian motion – 02.50.Ey Stochastic processes – 05.10.Gg Stochastic analysis methods (Fokker-Planck, Langevin, etc.)

The Fokker-Planck equation (FPE) in one dimension:

$$\frac{\partial P(x,t)}{\partial t} = \frac{\partial}{\partial x}\left[A(x,t)P(x,t)\right] + \frac{\partial^2}{\partial x^2}\left[B(x,t)P(x,t)\right], \tag{1}$$

which occupies a classic place in the field of "Brownian motion and fluctuations", is widely used as a tool for modelling various stochastic processes in many areas of physics, chemistry, biology, engineering and finance [1–3]. As is well known, while the stationary solution of the FPE can be given in closed form (at least up to quadratures) if the condition of detailed balance holds, the study of its time-dependent solution is, however, a much more complicated problem. In fact, exactly solvable FPE with an external time-dependent potential are extremely rare. In this communication we apply the Lie algebraic method to derive the propagator of the FPE:

$$\frac{\partial P(x,t)}{\partial t} = \left\{Bx\frac{\partial^2}{\partial x^2} + \left[C(t)x + D\right]\frac{\partial}{\partial x} + C(t)\right\}P(x,t), \tag{2}$$

and investigate the time evolution of the solution. Here $x \geq 0$, $B > 0$, and $C(t) > 0$. This equation represents the well-known "square-root" model of option pricing with time-dependent parameters in the field of finance [4]. In this model we have a variable diffusion coefficient and a time-dependent mean-reverting force. Such a model can be useful to study the problem of a Brownian walker with a linearly space-dependent diffusion coefficient, which could be realized experimentally by trapping particles between two nearly parallel walls [5]. Introducing a simple change of variables: $y = \sqrt{x}$, equation (2) can be recast in the following form:

$$\frac{\partial u(y,t)}{\partial t} = \left\{\frac{1}{4}B\frac{\partial^2}{\partial y^2} + \frac{1}{2}\left[C(t)y + \frac{2D+3B}{2y}\right]\frac{\partial}{\partial y} + \frac{1}{2}C(t) - \frac{2D+3B}{4y^2}\right\}u(y,t) \equiv H(t)u(y,t), \tag{3}$$

where $u(y,t) = yP(x,t)$. This FPE represents a generalization of the well-known Rayleigh process, which involves a constant diffusion coefficient and a time-dependent anharmonic oscillator potential $V(y,t) = \frac{1}{4}[C(t)y^2 + (2D + 3B)\ln y]$ [2]. This particular choice of potential can be used to model the behaviour of several physical and biological systems, *e.g.* the study of neuron models [6], stochastic resonance in monostable nonlinear oscillators [7] and its possible application to spatially extended systems [8]. Furthermore, the knowledge of the exact propagator of the model Fokker-Planck equation can be useful as a benchmark to test approximate numerical or analytical procedures.

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To begin with, we rewrite the operator $H(t)$ in terms of the su(1,1) generators as follows [9]:

$$H(t) = J_+ - C(t)J_0 - \frac{D+B}{2B}C(t), \qquad (4)$$

where

$$J_+ = \frac{1}{4}B\frac{\partial^2}{\partial y^2} + \frac{2D+3B}{4y}\frac{\partial}{\partial y} - \frac{2D+3B}{4y^2}$$
$$J_- = \frac{1}{B}y^2 \quad , \quad J_0 = -\frac{1}{2}\left(y\frac{\partial}{\partial y} + \frac{1}{2} + \frac{2D+3B}{2B}\right) \cdot \qquad (5)$$

The operators J_+, J_0 and J_- form the Lie algebra su(1,1):

$$[J_+, J_-] = -2J_0 \quad , \quad [J_0, J_\pm] = \pm J_\pm. \qquad (6)$$

One may define the evolution operator $U(t,0)$ such that

$$u(y,t) = U(t,0)u(y,0). \qquad (7)$$

Inserting equation (7) into equation (3) yields the evolution equation

$$H(t)U(t,0) = \frac{\partial}{\partial t}U(t,0) \quad , \quad U(0,0) = 1. \qquad (8)$$

Assuming that $U(t,0)$ takes the form:

$$U(t,0) = U_I(t,0)\exp\left[-\frac{B+D}{2B}\int_0^t C(\tau)\ \mathrm{d}\tau\right], \qquad (9)$$

the evolution equation in equation (8) is reduced to

$$H_I(t)U_I(t,0) = \frac{\partial}{\partial t}U_I(t,0) \quad , \quad U_I(0,0) = 1, \qquad (10)$$

with $H_I(t)$ being defined by

$$H_I(t) = J_+ - C(t)J_0. \qquad (11)$$

Since the su(1,1) algebra is a real "split three-dimensional" simple Lie algebra, the Wei-Norman theorem states that the evolution $U_I(t,0)$ can be expressed in the following form [10]:

$$U_I(t,0) = \exp\left[c_3(t)J_-\right]\exp\left[c_2(t)J_0\right]\exp\left[c_1(t)J_+\right] \qquad (12)$$

where $c_i(t)$ are to be determined. Then by direct differentiation with respect to time, we obtain

$$\frac{\partial}{\partial t}U_I(t,0) = [h_+(t)J_+ + h_0(t)J_0 + h_-(t)J_-]U_I(t,0) \qquad (13)$$

with

$$h_+(t) = \exp(c_2)\frac{\mathrm{d}c_1}{\mathrm{d}t},$$
$$h_0(t) = \frac{\mathrm{d}c_2}{\mathrm{d}t} + 2c_3\exp(c_2)\frac{\mathrm{d}c_1}{\mathrm{d}t},$$
$$h_-(t) = \frac{\mathrm{d}c_3}{\mathrm{d}t} + c_3\frac{\mathrm{d}c_2}{\mathrm{d}t} + c_3^2\exp(c_2)\frac{\mathrm{d}c_1}{\mathrm{d}t} \cdot \qquad (14)$$

Substituting equations (11, 12) and (13) into equation (10), and comparing the two sides, we obtain after simplification

$$\frac{\mathrm{d}c_3(t)}{\mathrm{d}t} = c_3^2(t) + C(t)c_3(t) \quad , \quad c_3(0) = 0, \qquad (15)$$
$$c_2(t) = -\int_0^t \left[2c_3(\tau) + C(\tau)\right]\ \mathrm{d}\tau, \qquad (16)$$
$$c_1(t) = \int_0^t \exp\left[-c_2(\tau)\right]\ \mathrm{d}\tau. \qquad (17)$$

Equation (15), which is just a Bernoulli equation, is the equation we have to solve first to determine $c_3(t)$, and obviously the only admissible solution is the trivial solution $c_3(t) = 0$. Once $c_3(t)$ is determined, $c_1(t)$ and $c_2(t)$ can be obtained readily by direct integration:

$$c_2(t) = -\int_0^t C(\tau)\ \mathrm{d}\tau \quad ,$$
$$c_1(t) = \int_0^t \exp\left[-c_2(\tau)\right]\ \mathrm{d}\tau. \qquad (18)$$

Hence, we have obtained an exact form of the time evolution operator $U(t,0)$ of the FPE in equation (3):

$$U(t,0) = \exp\left[-\frac{c_2(t)}{2}\right]\exp\left[-\frac{c_2(t)}{2}y\frac{\partial}{\partial y}\right]\exp\left[c_1(t)J_+\right]. \qquad (19)$$

Without loss of generality, we suppose that $u(y,0) = y^{(\alpha+1)/2}v(y,0)$, where $\alpha = -(2D+3B)/B$ and $v(y,0)$ is defined in terms of the Fourier-Bessel integral:

$$v(y,0) = \int_0^\infty \mathrm{d}\nu\ \nu J_{(\alpha-1)/2}(y\nu)$$
$$\times \int_0^\infty \mathrm{d}y'\ y' J_{(\alpha-1)/2}(y'\nu)v(y',0), \qquad (20)$$

for $\alpha > 0$[1]. Then it is not difficult to show that $u(y,t)$ is given by

$$u(y,t) = \int_0^\infty \mathrm{d}y' K(y,t;y',0)\ u(y',0) \qquad (21)$$

[1] Such an expansion of $v(y,0)$ is valid at every continuity point of $v(y,0)$ provided that:

1. The function $v(y,0)$, defined in the semi-infinite interval $(0,\infty)$, is piecewise continuous and of bounded variation in every finite subinterval $[y_1,y_2]$, where $0 < y_1 < y_2 < \infty$;
2. The integral
$$\int_0^\infty \sqrt{y}\ |v(y,0)|\mathrm{d}y$$
is finite.

See, for example, the book *"Special Functions & Their Applications"* by N.N. Lebedev (Dover Publications Inc., N.Y., 1972).

with

$$K(y,t;y',0) = \left\{ y'^{1-\alpha} y^{1+\alpha} \exp\left[-\frac{\alpha+3}{2} c_2(t) \right] \right\}^{1/2} \times \int_0^\infty d\nu\, \nu \exp\left[-\frac{B\, c_1(t)}{4} \nu^2 \right] J_{(\alpha-1)/2}\left(y'\nu \right) \times J_{(\alpha-1)/2}\left(y\, \nu \exp[-c_2(t)/2] \right). \quad (22)$$

The function J_μ is the Bessel function of the first kind of order μ. Here we have made use of the fact that $y^{(\alpha+1)/2} J_{(\alpha-1)/2}(y\nu)$ is an eigenfunction of the operator J_+ with the eigenvalue $-B\nu^2/4$, as well as the well-known relation

$$\exp\left(\eta y \frac{\partial}{\partial y} \right) f(y) = f\left(y \exp(\eta) \right). \quad (23)$$

The integral over ν can be evaluated to give [11]

$$\frac{2}{B\ c_1(t)} \exp\left\{ -\ \frac{y'^2 + y^2 \exp[-\ c_2(t)]}{B\ c_1(t)} \right\} \times I_{(\alpha-1)/2}\left(\frac{2\, y'\, y \exp[-c_2(t)/2]}{B\ c_1(t)} \right) \quad (24)$$

for $(\alpha-1)/2 > -1$, $y' > 0$ and $y \exp[-c_2(t)/2] > 0$. The function I_μ is the modified Bessel function of the first kind of order μ. The desired propagator $K(y,t;y',0)$ is thus found to be

$$K(y,t;y',0) = \frac{2}{Bc_1(t)} \left\{ y'^{1-\alpha} y^{1+\alpha} \exp\left[-\frac{\alpha+3}{2} c_2(t) \right] \right\}^{1/2} \times \exp\left\{ -\frac{y'^2 + y^2 \exp[-c_2(t)]}{Bc_1(t)} \right\} \times I_{(\alpha-1)/2}\left(\frac{2y'y \exp(-c_2(t)/2)}{Bc_1(t)} \right). \quad (25)$$

Consequently, assuming that $u(y,0) = \delta(y-y_0)$, the time evolution of the random particle is described by the propagator $K(y,t;y_0,0)$. To calculate the total probability of finding the random particle within the interval $[0,\infty)$ at any time $t > 0$, we simply need to evaluate the integral:

$$\int_0^\infty \mathrm{d}y\, K(y,t;y_0,0) = \frac{2}{Bc_1(t)} \left\{ y'^{1-\alpha} \exp\left[-\frac{\alpha+3}{2} c_2(t) \right] \right\}^{1/2} \exp\left(-\frac{y'^2}{Bc_1(t)} \right) \times \int_0^\infty dy\ y^{(1+\alpha)/2} \exp\left\{ -\frac{y^2 \exp[-c_2(t)]}{Bc_1(t)} \right\} \times I_{(\alpha-1)/2}\left(\frac{2y'y \exp(-c_2(t)/2)}{Bc_1(t)} \right). \quad (26)$$

For $(\alpha-1)/2 > -1$, $y' > 0$, the integral over y on the right-hand side yields [11]

$$\left\{ \frac{2}{Bc_1(t)} \left\{ y'^{1-\alpha} \exp\left[-\frac{\alpha+3}{2} c_2(t) \right] \right\}^{1/2} \exp\left(-\frac{y'^2}{Bc_1(t)} \right) \right\}^{-1}. \quad (27)$$

Hence, the total probability of finding the random particle within the interval $[0,\infty)$ is always conserved, *i.e.*

$$\int_0^\infty \mathrm{d}y\, K(y,t;y_0,0) = 1. \quad (28)$$

Finally, it is not difficult to show that the solution of the FPE in equation (2) is given by

$$P(x,t) = \int_0^\infty \mathrm{d}x' G(x,t;x',0)\ P(x',0) \quad (29)$$

with

$$G(x,t;x',0) = \frac{1}{Bc_1(t)} \left\{ \left(\frac{x}{x'} \right)^{(\alpha-1)/2} \exp\left[-\frac{\alpha+3}{2} c_2(t) \right] \right\}^{1/2} \times \exp\left\{ -\frac{x' + x \exp[-c_2(t)]}{Bc_1(t)} \right\} \times I_{(\alpha-1)/2}\left(\frac{2\sqrt{x'\, x} \exp(-c_2(t)/2)}{Bc_1(t)} \right). \quad (30)$$

Beyond question, the total probability of finding the Brownian walker with a space-dependent diffusion coefficient within the interval $[0,\infty)$ is always conserved too.

Furthermore, we can generalize the solution given in equations (21, 22) to the case which has a moving barrier at $y > 0$ in addition to the fixed barrier at $y = 0$ as follows:

$$u(y,t) = \int_0^L \mathrm{d}y' \mathcal{K}(y,t;y',0)\ u(y',0) \quad (31)$$

with

$$\mathcal{K}(y,t;y',0) = \sum_{n=1}^\infty \frac{2y'}{L^2 J_{\omega+1}^2(x_{\omega n})} \left\{ \frac{y}{y'} \exp\left[-\frac{c_2(t)}{2} \right] \right\}^{\omega+1} \times J_\omega\left(\frac{x_{\omega n}}{L} y \exp\left[-\frac{c_2(t)}{2} \right] \right) J_\omega\left(\frac{x_{\omega n}}{L} y' \right) \times \exp\left[-\frac{Bc_1(t)}{4L^2} x_{\omega n}^2 \right]. \quad (32)$$

for $\omega \equiv (\alpha-1)/2 > -1$. Here $x_{\omega n}$ denotes the nth zero of the Bessel function J_ω, and L is the position of the moving barrier at $t = 0$. It is not difficult to show that at $t \geq 0$ the kernel $\mathcal{K}(y,t;y',0)$ will disappear at $y = L \exp[c_2(t)/2]$. That is, the moving barrier is moving along the trajectory $y^*(t) = L \exp[c_2(t)/2]$. Accordingly, the corresponding solution of the FPE in equation (2), is simply given by

$$P(x,t) = \int_0^{\sqrt{L}} \mathrm{d}x' \mathcal{G}(x,t;x',0)\ P(\sqrt{x'},0) \quad (33)$$

where the propagator $\mathcal{G}(x,t;x',0)$ is defined as

$$\mathcal{G}(x,t;x',0) = \frac{1}{2\sqrt{x}}\mathcal{K}(\sqrt{x},t;\sqrt{x'},0). \qquad (34)$$

It should be noted that such a system is bounded by two barriers, namely a fixed barrier at $x = 0$ and a moving barrier along the trajectory $x^*(t) = L^2 \exp[c_2(t)]$, and hence it could be useful for the general problem of a Brownian walker with a space-dependent diffusion coefficient, which is trapped between two parallel plates.

In summary, we have investigated the algebraic structure of the Fokker-Planck equation with a variable diffusion coefficient and a time-dependent mean-reverting force. Such a model could be useful to study the general problem of a Brownian walker with a linearly space-dependent diffusion coefficient. We also show that this model is related to the Fokker-Planck equation with a constant diffusion coefficient and a time-dependent anharmonic potential of the form $V(x,t) = \frac{1}{2}a(t)x^2 + b\ln x$, which has been widely applied to model different physical and biological phenomena, *e.g.* the study of neuron models and stochastic resonance in monostable nonlinear oscillators. Using the Lie algebraic approach we have derived the exact diffusion propagators for these two types of Fokker-Planck equations. These exact diffusion propagators not only enable us to study the time evolution of the corresponding stochastic systems, but the knowledge of these exact propagators can also be useful as a benchmark to test approximate numerical or analytical procedures. Furthermore, the Lie algebraic method is very simple and could be easily extended to the more general Fokker-Planck equations with well-defined algebraic structures.

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